CALCULATION OF A LAMINAR AXIAL FLOW OF A NONLINEAR VISCOPLASTIC MEDIUM IN AN ANNULAR CHANNEL

Z. P. Shul'man

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We calculate the fundamental field and integral characteristics (velocity profiles, tangential stresses, outflow rate, pressure loss, boundaries of the quasirigid core) for a nonlinear viscoplastic Cesson medium in stationary stabilized flow in a coaxially cylindrical channel.

In many technological processes we encounter problems involving the motion of viscoplastic media in annular channels. As examples we cite "tube-in-tube" type heat exchangers, extrusion and screw-type apparatuses, and, finally, drilling assemblies in which special clay and cement solutions are pumped through the annular gap between the well and tower tubes. Without these solutions efficient passage of oil, gas, or water through apertures would not be possible.

Until recently, the mechanical behavior of the overwhelming majority of viscoplastic fluid dispersions was described by the linear rheological Schwedoff–Bingham equation

$$\tau = \tau_0 + \mu_p \dot{\gamma}. \tag{1}$$

Here τ is the uniaxial shear stress; τ_0 is the flow limit; $\dot{\gamma} = du/dn$ is the rate of shear; μ_p is the plastic viscosity.

Thanks to progress in rheometry a limited applicability of Eq. (1) has been established. In actuality, the flow curves of viscoplastic compositions, to one degree or another, are nonlinear in regions of small and moderate values of $\dot{\gamma}$.

Based on the treatment of a large amount of experimental data, obtained by various authors, a generalized rheological equation for a nonlinear viscoplastic medium was formulated in [1, 2]:

$$\tau^{\frac{1}{n}} = \tau_0^{\frac{1}{n}} + (\mu_p \dot{\gamma})^{\frac{1}{m}}. \tag{2}$$

Here m and n are nonlinearity parameters of the flow curve (m > 0; n > 0), not necessarily integers.

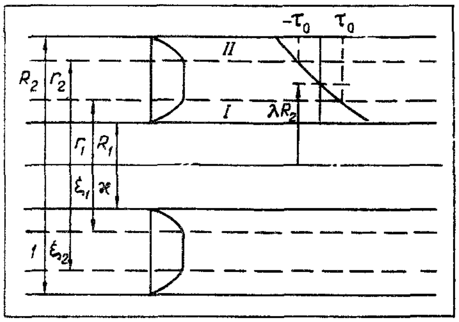

Fig. 1. Flow diagram.

The four-parameter model (2) unifies in a nonlinear manner the viscoplastic and anomaloviscous properties of a medium. It generalizes the known rheological equations of state: Newtonian (m = n, $\tau_0 = 0$), Ostwald de Villat ($\tau_0 = 0$), Schwedoff–Bingham (m = n = 1), Bulkley–Herschel (n = 1), Cesson (m = n = 2), and Ruffers (m = n).

The problem of the flow of a viscoplastic medium in an annular channel, even in the case of a linear Schwedoff–Bingham medium, has not been solved analytically to date. The widely used equation due to Fredricson and Bird [3] appears in the form of a set of tables and graphs with the aid of which the necessary characteristics (pressure, outflow rate, velocity profile, and rheodynamic losses) can be rapidly calculated.

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We give below a solution of the same problem for a nonlinear viscoplastic liquid, where m = n = 2 (Cesson medium)

$$\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{\mu_p \dot{\gamma}}. \tag{3}$$

We consider a flow in the direction of the positive z axis through an infinite annular channel formed by two coaxial cylinders of radii R_1 and R_2, under the action of a constant pressure gradient $A = |dp/dz|$. In terms of stresses the equation of motion then has the form

$$\frac{1}{r}\frac{d}{dr}(\tau r) = -A \tag{4}$$

or, after a single integration,

$$\tau = -\frac{A}{2} r + \frac{c}{r} \tag{5}$$

(c is a constant of integration).

Thus the tangential stress profile (5) across the annular gap is conserved with respect to the rheological properties of the medium.

If the channel boundaries are fixed, then there exists in the interior of the channel a cylindrical surface $r = \lambda R_2$ ($\lambda > 0$), where the shear stress is equal to zero. Then

$$\tau = \frac{A}{2}\left[\frac{(\lambda R_2)^2}{r} - r\right]. \tag{6}$$

The magnitude of the constant λ remains to be determined. The existence of a minimum on the profile $\tau(r)$ necessarily implies the existence in the annular region of two cylindrical surfaces $r = r_1$ and $r = r_2$, on which the shear stress assumes the values $+\tau_0$ and $-\tau_0$, respectively (Fig. 1). In this region, $r_1 < r < r_2$, the tangential stresses are less than the flow limit and the medium moves in the z direction as a quasirigid rod. On both sides of this region the velocity gradients $\dot{\gamma}$ are of opposite signs: positive in the tube interior (zone I) and negative at the exterior part of the tube (zone II).

Upon substituting $\dot{\gamma}$ from Eq. (3) into Eq. (6), taking account of the sign, transforming to dimensionless variables, and performing a single integration with the "no slip" boundary conditions at both walls taken into account, we obtain the velocity field

$$w_1(\xi) = \beta(\xi - \varkappa) + \lambda^2 \ln\frac{\xi}{\varkappa} - \frac{1}{2}(\xi^2 - \varkappa^2) - 2\beta^{\frac{1}{2}}[f_1(\xi) - f_1(\varkappa)] \tag{7a}$$

$$(\varkappa \leqslant \xi \leqslant \xi_1),$$

$$w_2(\xi) = \beta(1 - \xi) + \lambda^2 \ln \xi + \frac{1}{2}(1 - \xi^2) + 2\beta^{\frac{1}{2}}[f_2(\xi) - f_2(1)] \tag{7b}$$

$$(\xi_2 \leqslant \xi \leqslant 1),$$

$$w_0 = w_1(\xi_1) = w_2(\xi_2) \quad (\xi_1 \leqslant \xi \leqslant \xi_2). \tag{7c}$$

Here

$$w = \frac{2\mu_p}{AR_2^2}u, \quad \beta = \frac{2\tau_0}{AR_2}, \quad \varkappa = \frac{R_1}{R_2};$$

$$\xi = \frac{r}{R_2}, \quad \xi_1 = \frac{r_1}{R_2}, \quad \xi_2 = \frac{r_2}{R_2}; \tag{8}$$

$$f_1(\xi) = \int\left(\frac{\lambda^2}{\xi} - \xi\right)d\xi; \quad f_2(\xi) = \int\left(\xi - \frac{\lambda^2}{\xi}\right)d\xi. \tag{9}$$

We have the following conditions for determining the unknown quantities:

a) the equation for the velocities at both boundaries of the quasirigid core:

$$w_1(\xi_1) = w_2(\xi_2).$$

Using Eqs. (7a) and (7b), we obtain the algebraic equation

$$\beta(1-\xi_1-\xi_2+\varkappa)+\lambda^2\ln\frac{\xi_2\varkappa}{\xi_1}+\frac{1}{2}(1-\xi_2^2+\xi_1^2-\varkappa^2)+2\beta^{\frac{1}{2}}\left[f_2(\xi_2)-f_1(1)+f_1(\xi_1)-f_1(\varkappa)\right]=0; \tag{10}$$

b) the equilibrium balance of forces acting on a cylindrical element of the core of length L; herein we denote the pressures on the end surfaces of the element by p_1 and p_2 ($p_2 > p_1$)

$$(p_1-p_2)(\pi r_2^2-\pi r_1^2)=\tau_0(2\pi r_2-2\pi r_1)L$$

or, after obvious simplifications,

$$\xi_2-\xi_1=\beta. \tag{11}$$

Thus the plasticity parameter may be interpreted as the dimensionless width of the quasirigid flow core;

c) the flow conditions at the boundaries of the quasirigid core, namely,

$$\tau(\xi_1)=\tau_0 \text{ and } \tau(\xi_2)=-\tau_0.$$

Using Eq. (6) we find

$$\lambda^2=\xi_1\xi_2. \tag{12}$$

Equations (10)-(12) form a compatible system of equations for determining the unknown parameters λ, ξ_1, ξ_2.

As a result we arrive at a complicated transcendental equation for ξ_2 or ξ_1, which is not solvable in analytic form. Approximate methods were presented in [4] for finding the boundaries of the quasirigid core in the case of comparatively small values of β, i.e., when ξ_1 and ξ_2 are close to λ_0, the relative radial coordinate of the maximum on the velocity profile of a Newtonian liquid in the annular channel.

We consider another method of calculating the fundamental characteristics of the motion, a method similar in concept to that used by Laird [5]. From the condition (10) for the equation of velocities at both boundaries of the quasirigid core, and also from the relation (12), we arrive at the equation

$$\xi_1\xi_2=\left\{\beta(1-\xi_1-\xi_2+\varkappa)+\frac{1}{2}(1+\xi_1^2-\xi_2^2-\varkappa^2)-2\beta^{\frac{1}{2}}\left[F_2(\xi_2;1)-F_1(\varkappa;\xi_1)\right]\right\}\left\{\ln\frac{\xi_1}{\varkappa\xi_2}\right\}^{-1}. \tag{10a}$$

We rewrite Eq. (10a) in the form

$$\Phi(\xi_1;\xi_2)=\xi_1\xi_2-\left\{\beta(1+\varkappa-\xi_1-\xi_2)+\frac{1}{2}(1-\varkappa^2+\xi_1^2-\xi_2^2)\right.$$
$$\left.-2\beta^{\frac{1}{2}}\left[F_2(\xi_2;1)-F_1(\varkappa;\xi_1)\right]\right\}\left\{\ln\frac{\xi_1}{\varkappa\xi_2}\right\}^{-1}=0. \tag{13}$$

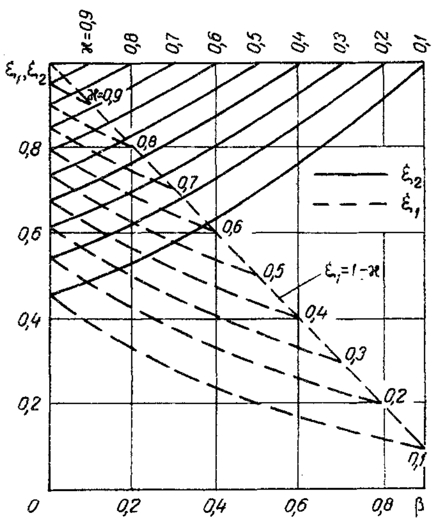

Fig. 2. Boundaries of quasirigid flow core, ξ_1 and ξ_2, versus the plasticity parameter β.

We solved Eq. (13) accurately by a numerical method with the aid of the Minsk-22 electronic digital computer. In addition, for each fixed value of $\varkappa$ and β, a sequence of arbitrarily selected values of $\alpha = \xi_1/\xi_2$ was given. From the condition $\varkappa \le \xi_1 \le \xi_2 \le 1$, it is obvious that $\varkappa \le \alpha \le 1$. In Eq. (13) the substitution of $\alpha\xi_2$ for ξ_1 transforms Eq. (13) to an algebraic equation in the one unknown ξ_2. The order of its solution is the following.

Since $\xi_2 \le 1$, then of all the roots of the equation $\Phi(\alpha;\xi_2) = 0$ only the positive real root located in the interval ($\varkappa$; 1) is of interest to us. We divide this interval into s equal parts and form the products

$$\Phi(x_{k-1})\,\Phi(x_k),\quad \text{where } k=0;\ 1,\ 2,\ \ldots,\ s. \tag{14}$$

We may determine the interval $\Delta x_k = x_k - x_{k-1}$, where the root is located from the sign of this product. We then divide Δx_k into

TABLE 1. Values of ξ_2, β, w_0, q, and q/β for Different Values of A and $\varkappa$

$\varkappa$	quantity	α = 0,15	0,20	0,25	0,30	0,35	0,40	0,45	0,50	0,55	0,60	0,65	0,70	0,75	0,80	0,85	0,90	0,95
0,1	ξ_2	0,8765	0,7950	0,7360	0,6905	0,6535	0,6230	0,5975	0,5755	0,5560	0,5390	0,5240	0,5110	0,4990	0,4885	0,4790	0,4705	0,4640
	β	0,745	0,6360	0,5520	0,4834	0,4248	0,3738	0,3286	0,2878	0,2502	0,2156	0,1834	0,1533	0,1248	0,0977	0,0718	0,0470	0,0232
	$w_0 \cdot 10^2$		0,1357	0,3317	0,6044	0,9512	1,3658	1,8456	2,3980	3,0347	3,7566	4,5785	5,5115	6,6084	7,8974	9,4762	11,512	14,398
	$q \cdot 10^4$	1,0000	6,0000	14,000	25,0	38,0	54,0	72,0	91,0	114,0	139,0	166,0	197,0	233,0	274,0	324,0	387,0	474,0
	$q/\beta \cdot 10^4$	2,0	9,0	27,0	52,0	91,0	145,0	218,0	318,0	455,0	643,0	907,0	1287,0	1866,0	2807,0	4512,0	8210,0	20430,0
0,2	ξ_2			0,9205	0,8590	0,8095	0,7690	0,7345	0,7045	0,6790	0,6560	0,6360	0,6180	0,6015	0,5870	0,5735	0,5615	0,5515
	β			0,6904	0,6013	0,5262	0,4614	0,4040	0,3522	0,3056	0,2624	0,2226	0,1854	0,1504	0,1174	0,0860	0,0562	0,0276
	$w_0 \cdot 10^2$			0,0088	0,0564	0,1581	0,3186	0,5439	0,8403	1,2054	1,6590	2,2011	2,8563	3,6571	4,6286	5,8664	7,4958	9,8761
	$q \cdot 10^4$			0,4	2,0	7,0	13,0	22,0	33,0	47,0	63,0	82,0	104,0	131,0	163,0	202,0	253,0	324,0
	$q/\beta \cdot 10^4$			0,6	4,0	13,0	29,0	55,0	96,0	153,0	240,0	363,0	563,0	872,0	1385,0	2350	4508,0	11750,0
0,3	ξ_2					0,9400	0,8900	0,8485	0,8120	0,7810	0,7530	0,7285	0,7065	0,6865	0,6685	0,6520	0,6370	0,6240
	β					0,6110	0,5340	0,4667	0,4060	0,3514	0,3012	0,2550	0,2120	0,1716	0,1337	0,0978	0,0637	0,0312
	$w_0 \cdot 10^2$					0,0050	0,0352	0,1044	0,2257	0,4030	0,6525	0,9793	1,4022	1,9464	2,6440	3,5621	4,8215	6,7252
	$q \cdot 10^4$					0,2	1,0	4,0	9,0	16,0	25,0	36,0	51,0	69,0	91,0	120,0	158,0	214,0
	$q/\beta \cdot 10^4$					0,4	3,0	9,0	22,0	44,0	82,0	142,0	239,0	400,0	681,0	1224,0	2480,0	6842,0
0,4	ξ_2							0,9515	0,9095	0,8730	0,8405	0,8120	0,7860	0,7630	0,7420	0,7225	0,7055	0,6895
	β							0,5233	0,4548	0,3922	0,3362	0,2842	0,2358	0,1908	0,1484	0,1084	0,0706	0,0345
	$w_0 \cdot 10^2$							0,0035	0,0264	0,0830	0,1872	0,3495	0,5891	0,9204	1,3783	2,0201	2,9245	4,3845
	$q \cdot 10^4$							0,1	1,0	3,0	7,0	12,0	20,0	31,0	45,0	64,0	90,0	130,0
	$q/\beta \cdot 10^4$							0,3	2,0	8,0	20,0	53,0	87,0	163,0	305,0	595,0	1279,0	9722,0
0,5	ξ_2									0,9590	0,9225	0,8900	0,8610	0,8345	0,8165	0,7890	0,7690	0,7510
	β									0,4315	0,3690	0,3115	0,2583	0,2086	0,1621	0,1183	0,0769	0,0376
	$w_0 \cdot 10^2$									0,0030	0,0230	0,0756	0,1760	0,3449	0,6074	1,0009	1,6139	2,6454
	$q \cdot 10^4$									0,1	0,8	2,0	6,0	11,0	18,0	29,0	46,0	71,0
	$q/\beta \cdot 10^4$									0,2	2,0	8,0	22,0	52,0	114,0	245,0	592,0	1895,0
0,6	ξ_2											0,9640	0,9320	0,9025	0,8760	0,8520	0,8300	0,8095
	β											0,3374	0,2796	0,2256	0,1752	0,1278	0,0830	0,0405
	$w_0 \cdot 10^2$											0,0029	0,0227	0,0783	0,1913	0,3938	0,7472	1,4110
	$q \cdot 10^4$											0,09	0,7	2,0	5,0	10,0	18,0	33,0
	$q/\beta \cdot 10^4$											0,3	2,0	10,0	29,0	79,0	221,0	814,0
0,7	ξ_2													0,9685	0,9390	0,9125	0,8885	0,8660
	β													0,2421	0,1878	0,1369	0,0888	0,0433
	$w_0 \cdot 10^2$													0,0030	0,0260	0,0946	0,2506	0,6033
	$q \cdot 10^4$													0,07	0,6	2,0	5,0	11,0
	$q/\beta \cdot 10^4$													0,027	3,0	15,0	56,0	262,0
0,8	ξ_2															0,9715	0,9450	0,9205
	β															0,1457	0,0945	0,0460
	$w_0 \cdot 10^2$															0,0039	0,0372	0,0163
	$q \cdot 10^4$															0,06	0,5	2,0
	$q/\beta \cdot 10^4$															0,4	6,0	36,0
0,9	ξ_2																	0,9740
	β																	0,0487
	$w_0 \cdot 10^2$																	0,0083
	$q \cdot 10^4$																	0,05
	$q/\beta \cdot 10^4$																	1,0

m equal parts and perform the calculations, comparing the values obtained with a preselected value δ, which defines the accuracy of the calculations. The process is repeated until the required precision is attained. In this way, for each value of α the dependence of ξ_2 and ξ_1 on the plastic parameter β and the relative channel aperture $\varkappa$ may be determined. The integrals appearing in Eq. (13) were obtained numerically in the process of finding the roots.

For the values obtained for ξ_2 and ξ_1 the pressure loss $\Delta p = AL$ was calculated from the condition (11) and the dimensionless outflow rate from the material balance

$$q = \int_{\varkappa}^{\xi_1} w_1(\xi)\,\xi d\xi + \frac{1}{2}(\xi_2^2 - \xi_1^2) w_0 + \int_{\xi_2}^{1} w_2(\xi)\,\xi d\xi \tag{15}$$

after performing the integration we obtained

$$q = \frac{\beta}{6}(1 - \xi_1^3 - \xi_2^3 + \varkappa^3) - \frac{\lambda^2}{4}(1 + \xi_1^2 - \xi_2^2 - \varkappa^2)$$

$$+ \frac{1}{8}(1 + \xi_1^4 - \xi_2^4 - \varkappa^4) + \beta^{\frac{1}{2}}\left[2\varphi_2(1) - 2\varphi_2(\xi_2) - 2\varphi_1(\xi_1) + 2\varphi_1(\varkappa) + \xi_2^2 f_2(1) - \xi_2^2 f_2(\xi_2) + \xi_1^2 f_1(\xi_1) - \xi_1^2 f_1(\varkappa)\right]. \tag{16}$$

Here

$$q = \frac{\mu_p Q}{\pi A R_2^2}; \quad \varphi_1(\xi) = \int f_1(\xi)\,\xi d\xi; \quad \varphi_2(\xi) = \int f_2(\xi)\,\xi d\xi. \tag{17}$$

The computer results are presented graphically in Figs. 2-4, giving the dependences of q, q/β, ξ_1, ξ_2 and w_0 on $\varkappa$ and β. In Figs. 2-4 the corresponding curves for a linear viscoplastic Schwedoff–Bingham body ($m = n = 1$) are shown by the dashed lines.

The dependence of the dimensionless coordinates of the quasirigid core on the viscoplasticity parameter β is shown in Fig. 2; this dependence was found to be a universal one for all $n = m$ by virtue of the conservative property of the basic relations (11) and (12).

For the various $\varkappa$ values the curves ξ_2 exhibit reverse convexity in their ascending branches, their ends being bounded by the line $\xi_2 = 1$. The curves ξ_1 (their descending portions) undergo a change in convexity downwards, i.e., towards the β axis. The geometric locus of the ends of these curves, their lower boundaries, is the straight line $\xi_1 = 1-\varkappa$. For an arbitrary fixed value of $\varkappa$ the difference of the ordinates of both branches is the value β. As the graph shows, with an increase in β the quasirigid flow region becomes wider. This graph makes it very simple to determine the resistance (pressure loss) in the channel for known values of $\varkappa$, ξ_2, μ_p and τ_0.

Of considerable interest is the graph showing the dependence of the longitudinal speed w_0 of the quasirigid core on the parameters β and $\varkappa$ (Fig. 3a). With an increase in β (we have in mind an increase in τ_0 for A = const, i.e., with an invariable moving pressure drop) the quantity w_0 decreases progressively to a value close to zero; the higher the value of $\varkappa$ the sooner this occurs. For comparison we have shown on the graph the corresponding curves for the linear Schwedoff–Bingham model. We note that for identical values of $\varkappa$ and for fixed β the speed w_0 is somewhat higher for the Schwedoff–Bingham medium. The other circumstance of importance consists in the fact that for small and moderate values of β the dependence $w_0(\beta)$ is close to linear, whereas for the Cesson medium the graphs are noticeably nonlinear over the whole range of possible β values. In addition, the quantity $w_{0,\,Ces}$ may be of an order less than the corresponding value for the Schwedoff–Bingham medium. Thus the nonlinearity of the rheological curve, although not very informative about the geometry of the flow (i.e., the values of ξ_1, ξ_2, λ), indicates the exceptionally strong influence on the kinematics and dynamics of the flow, and, in particular, on the speed of the quasirigid core.

The manner in which the outflow rate q or q/β through the annular gap depends on the plasticity parameter β (Fig. 3b, c) is of a similar nature. The influence of the rheological nonlinearity on q manifests itself here even more strongly than in the case of w_0. Thus, for example, for $\varkappa = 0.1$ and $\beta = 0.6$, we have

$$\frac{q_{SB}}{q_{ces}} \approx 12.$$

As $\varkappa$ increases this difference becomes less pronounced. For fixed β the channel geometry also effects the outflow rate unequally (Fig. 4). For a linear viscoplastic medium the values of q significantly exceed the

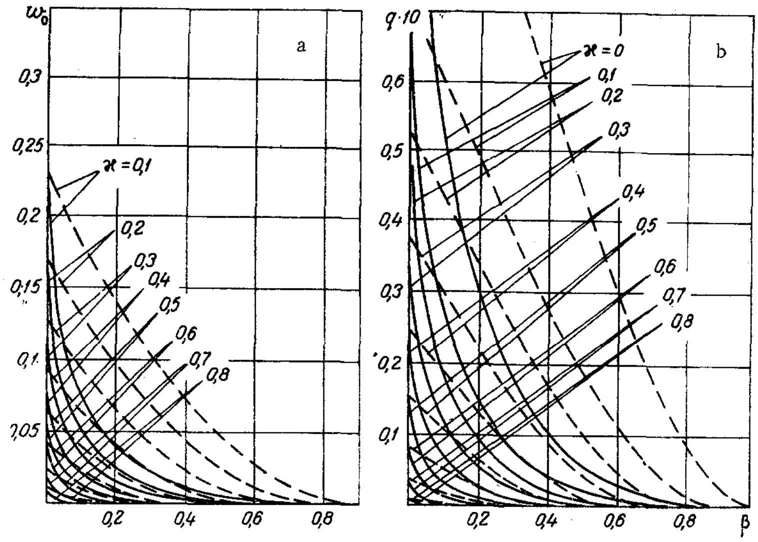

Fig. 3. a) Velocity w_0 of the quasirigid core as a function of the relative clogging of the channel $\varkappa$ and the plasticity parameter β; b) outflow rate q as a function of the parameters $\varkappa$ and β; c) plasticity parameter β as a function of the factor q/β. (Curves 1 and 2 correspond, respectively, to calculations using the Cesson and the Schwedoff–Bingham models.)

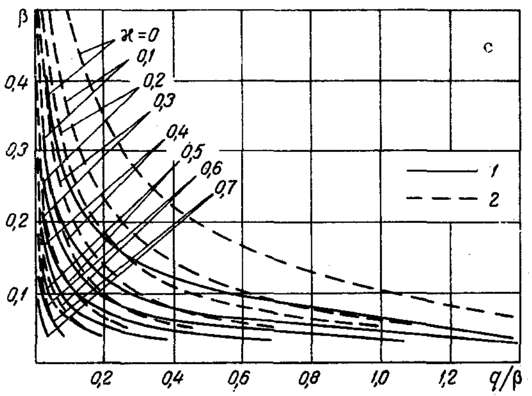

Fig. 4. Outflow rate q as a function of the channel geometry and the plasticity parameter β. (Curves 1 and 2 correspond, respectively, to calculations using the Cesson and the Schwedoff–Bingham models.)

corresponding values for its nonlinear analogue (Cesson model). The peculiarities inherent in the dependence $q(\varkappa)$ distinguish it from the dependence $q(\beta)$. As $\varkappa$ increases (a narrowing of the relative aperture), the outflow rate q naturally falls toward zero at $\varkappa = 1$. However in regions of small and moderate $\varkappa$ values (up to 0.15–0.20), the curves behave almost linearly for both the Schwedoff–Bingham and the Cesson media. Finally, the ratio q_{Sb}/q_{Ces} shows a weaker dependence on $\varkappa$ than on β.

Thus the calculations (see Table 1) exhibit the strong influence of the nonlinearity of the viscoplastic flow curve on the nature of the motion of the medium through the annular channel and on the integral characteristics of the flow (outflow rate, pressure).

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